The Lee Fields Medal III SOLUTIONS

TIME ALLOWED: UP TO THREE HOURS

TABLES AND CALCULATORS MAY BE USED.

ANSWER ALL TEN QUESTIONS

1. Does there exist a rectangle of area 401 with whole-number sides? Justify your answer.

   Solution: This is a trick question that was designed to make students think... is 401 a prime? If 401 is a prime, it cannot be written as the product of whole numbers. The task would be to prove that 401 is prime. How to do this? If 401 is not prime it has two prime factors, say m and n:
$$401 = m \cdot n.$$
   These factors cannot both be (strictly) larger than $\sqrt{401}$ otherwise:
$$m \cdot n > \sqrt{401}\sqrt{401} = 401,$$
   that is $m \cdot n$ would be too big. Note $\sqrt{401} \approx 20.02$ and so either m or n is a prime number smaller than 20. Without loss of generality[1], assume that m is a prime number smaller than 20. Then
$$m \in \{2, 3, 5, 7, 11, 13, 17, 19\}.$$
   We calculate $401/m$ for each of these and none give a whole number. Therefore none of these m is a factor, and hence 401 is prime.

   But of course 401 doesn't have to be non-prime or *composite* to have whole number factors. We have the wholly trivial:
$$401 = 401 \times 1.$$

2. A real number that can be written in the form
$$\frac{m}{n},$$
   with m and n whole numbers is a *rational* number, while one which cannot is called an *irrational* number. Examples of irrational numbers include $\sqrt{2}$, e, and π.

   (a) Given that $\sqrt{2}$ is irrational, show that $1 - \sqrt{2}$ is irrational.

   (b) Hence, or otherwise, write down two irrational numbers whose sum is rational.

[1] if m does not work, if $m > \sqrt{401}$, then instead we can make the same argument with n

Solution:

(a) We use a *proof by contradiction*. Suppose that $1-\sqrt{2}$ is *not* irrational. Then it is equal to a fraction of whole numbers:

$$1-\sqrt{2}=\frac{m}{n},$$

with $m,n\in\mathbb{Z}$. But this implies that

$$\sqrt{2}=1-\frac{m}{n}=\frac{n-m}{n}.$$

If $m,n\in\mathbb{Z}$, then so is $n-m$. So if $1-\sqrt{2}$ is not irrational, then $\sqrt{2}$ is rational. This is a contradiction, indicating that the original supposition that $1-\sqrt{2}$ is not irrational is false, and so $1-\sqrt{2}$ is irrational.

(b) For example, $\sqrt{2}$ and $1-\sqrt{2}$ are irrational and:

$$\sqrt{2}+1-\sqrt{2}=1=\frac{1}{1},$$

is rational.

3. Suppose that $m>n$ are whole numbers. Show that $m^2+n^2>2mn$. Furthermore, show that a triangle with side lengths m^2+n^2, m^2-n^2, $2mn$ is a right-angled-triangle.

Solution: The first part is a trick:

$$\begin{aligned}
&(m-n)^2>0\\
\implies\ & m^2-2mn+n^2>0\\
&\underset{+2mn}{\implies}\ m^2+n^2>2mn.
\end{aligned}$$

We can use the *converse*[2] of Pythagoras Theorem. We know that $m^2+n^2\geq m^2-n^2$ and from the first part $m^2+n^2\geq 2mn$. To be a triangle, we probably need $n>0$, and this forces $m^2+n^2>m^2-n^2$. So we have our hypotenuse $h=m^2+n^2$, and other sides $a=m^2-n^2$ and $b=2mn$. First we calculate the square of the hypotenuse:

$$\begin{aligned}
h^2&=(m^2+n^2)^2\\
&=m^4+2m^2n^2+n^4
\end{aligned}$$

and also:

$$\begin{aligned}
a^2+b^2&=(m^2-n^2)^2+(2mn)^2\\
&=m^4-2m^2n^2+n^4+4m^2n^2\\
&=m^4+2m^2n^2+n^4=h^2,
\end{aligned}$$

therefore the triangle with those side-lengths is a right-angled triangle.

[2]Pythagoras Theorem says: if Δ is a right-angled triangle, with side-lengths a,b and hypotenuse h, then $h^2=a^2+b^2$. The *converse* of Pythagoras Theorem is also true, and says if Δ is a triangle with side-lengths a,b,c, and $c^2=a^2+b^2$, then Δ is a right-angled triangle with hypothenuse c.

4. Consider the three lines:

$$\begin{aligned} \ell_1 : \quad & y = 3x + 1 \\ \ell_2 : \quad & y = x + 2 \\ \ell_3 : \quad & y = -2x + 6 \end{aligned}$$

Does the intersection $\ell_1 \cap \ell_2 \cap \ell_3$ contain any points? Justify your answer.

Solution: A rough sketch might help, because $y = mx + c$ is a line of slope m and y-intercept c... but in fact only shows us that the three intersections $\ell_1 \cap \ell_2$, $\ell_1 \cap \ell_3$, and $\ell_2 \cap \ell_3$ occur in the first quadrant.

Slightly better than brute force, is to find the intersections $\ell_1 \cap \ell_2$ and see is that point on ℓ_3 also. So $\ell_1 \cap \ell_2$:

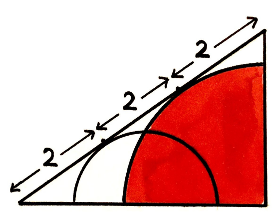
$$\begin{aligned} y &= 3x + 1 \\ y &= x + 2 \\ \implies 3x + 1 &= x + 2 \\ \implies 2x &= 1 \\ \implies x &= \frac{1}{2} \\ \implies y &= \frac{1}{2} + 2 = \frac{5}{2}, \end{aligned}$$

so $\ell_1 \cap \ell_2 = \{(1/2, 5/2)\}$. Is this point on ℓ_3?

$$-2x + 6 = -2(1/2) + 6 = 5 \neq \frac{5}{2} = y,$$

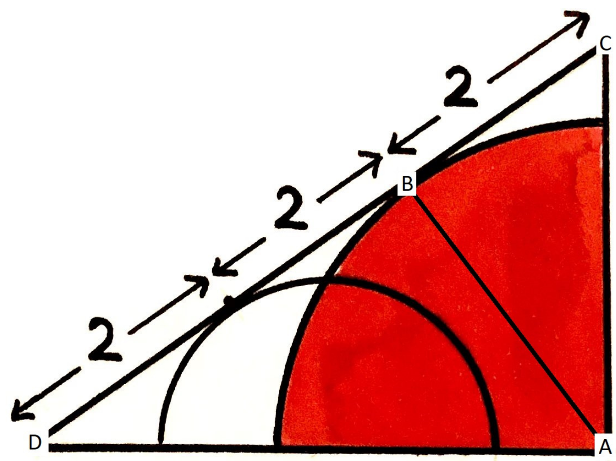
no. So $\ell_1 \cap \ell_2 \cap \ell_3$ is empty.

5. Find the area of the quarter-circle.



Solution: This question is thanks to `@Cshearer41` on Twitter. The quarter circle is a red herring.

Label as follows:



Note there is a theorem that a tangent, in this case $[CD]$, to a circle is perpendicular to the radius from the centre to the point of contact, in this case $[AB]$. This ensures that ΔABC and ΔABD are both right-angled triangles. Along with ΔACD. Denote $r := |AB|$, $x = |AC|$, and $y = |AD|$. That is we have three unknowns, so we probably need three equations. We can have three equations from three applications of Pythagoras Theorem, to ΔABC:

$$x^2 = r^2 + 2^2,$$

to ΔABD:

$$y^2 = r^2 + 4^2,$$

and the larger ΔACD:

$$6^2 = x^2 + y^2.$$

But we have both x and y in terms of r:

$$\begin{aligned} x^2 + y^2 &= r^2 + 4 + r^2 + 16 \overset{!}{=} 36 \\ \implies 2r^2 &= 16 \\ \underset{\div 2}{\implies} r^2 &= 8 \\ \underset{\sqrt{\ },r>0}{r} &= \sqrt{8} \end{aligned}$$

$$\implies A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(\sqrt{8})^2 = 2\pi$$

An alternative route to $r = \sqrt{8}$ was provided by the competition winner Krzysztof Przestrzelski. Consider angles $\angle BAD$ and $\angle ACB$. Because $\angle BAC = 90° - \angle ABD$, we have that $\angle BAD = \angle ACB$ and so they have the same tangent:

$$\begin{aligned}\tan(\angle BAD) &= \tan(\angle ACB)\\ \implies \frac{4}{r} &= \frac{r}{2}\\ \underset{\times 2r}{\Longrightarrow}\ 8 &= r^2\\ \underset{\sqrt{},\, r>0}{\Longrightarrow}\ r &= \sqrt{8}\end{aligned}$$

6. With the aid of a diagram, or otherwise, prove that for $0° < \theta < 90°$

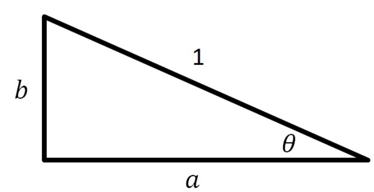
$$\sin\theta + \cos\theta > 1.$$

Solution: There are a number of approaches to this question. Perhaps the most natural is to take a right-angled triangle with hypotenuse one and angle θ:

If we call the sides by a and b, note we have:

$$\sin\theta = \frac{b}{1} \implies b = \sin\theta,$$

and similarly $a = \cos\theta$:



Now we have that the straight-line distance $h = 1$ is shorter than the distance:

$$a + b = \cos\theta + \sin\theta.$$

The competition winner Krzysztof Przestrzelski had an even better argument. Draw a right-angled triangle with angle θ. It has side lengths o, a and hypotenuse length h. As above,

$$o + a > h \underset{\div h}{\Longrightarrow} \frac{o}{h} + \frac{a}{h} > 1 \implies \sin\theta + \cos\theta > 1.$$

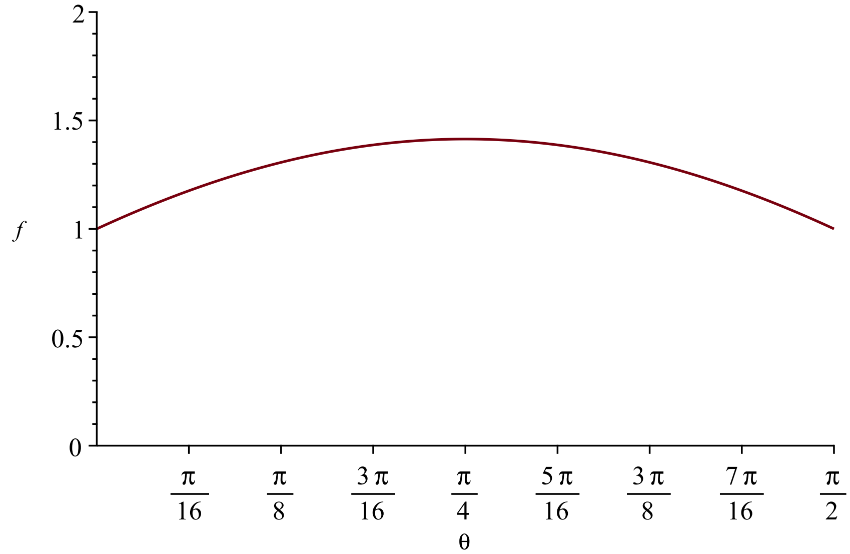
Another student Qian Kai Lim had an alternative approach. They defined a function:

$$f(\theta) = \sin\theta + \cos\theta.$$

They had[3] $f(0) = f(\pi/2) = 1$. They showed between $0 < \theta < \pi/2$, f had a single maximum, where $f' = 0$, at $\theta = \pi/4$; and also $f'' < 0$ on $0 < \theta < \pi/2$ implying that the curve has $\bigcap$ geometry. The curve could not go below $y = 1$ without having additional turning points.

[3] they needed radians as they used calculus

The student also produced a nice sketch of the curve $y = f(\theta)$:



7. Where d is the outcome of a rolled dice, consider the quadratic function:

$$q(x) = x^2 + 3x + d.$$

What is the probability that q has real roots?

Solution: When does a quadratic function have real roots? Let $f(x) = ax^2 + bx + c$. The roots of f are given by the "$-b$" formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

An issue occurs when $b^2 - 4ac < 0$, when we have the square root of a negative number, and we have complex roots. When $b^2 - 4ac \geq 0$, we have real roots.

For $q(x) = x^2 + 3x + d$, it will have real roots when:

$$b^2 - 4ac = 3^2 - 4(1)(d) = 9 - 4d \geq 0 \underset{+4d}{\Longrightarrow} 4d \leq 9 \underset{\div 4}{\Longrightarrow} d \leq \frac{9}{4} = 2.25.$$

As the outcome of the roll of a dice, $d \in \{1, 2, 3, 4, 5, 6\}$, and so we have real roots if $d \in \{1, 2\}$. Of course, therefore

$$\mathbb{P}[\,q \text{ has real roots}] = \mathbb{P}[d \in \{1, 2\}] = \frac{2}{6} = \frac{1}{3}.$$

8. A student is in the center of a square pool, the teacher at the corner. Teacher runs three times as fast as student swims, but the student runs faster than the teacher. Can the student escape past the teacher?

Solution: Let us set up some notation first of all. Let the pool have side-length $2L$. Let the speed of the student be v and the speed of the teacher be $3v$. Let t_S be the time it takes the student to enact a plan, and t_T the time it takes the teacher to chase. Note assuming constant speed, where s is distance:

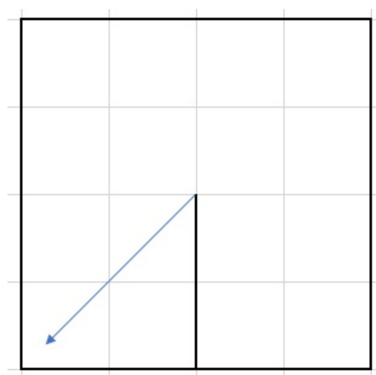
$$s = vt \implies t = \frac{s}{v}.$$

Define

$$t_0 = \frac{L}{v}.$$

We assume that the teacher always starts in the top-right corner.

(a) *Corner Approach* In this approach the student heads to the furthest point from the teacher:



To find the distance s_S the student swims we form a right-angled triangle with the arrow:

$$s_S^2 = L^2 + L^2 = 2L^2 \underset{\sqrt{},\, s_S>0}{\implies} s_S = \sqrt{2}L.$$

Therefore the student can get to the corner in time:

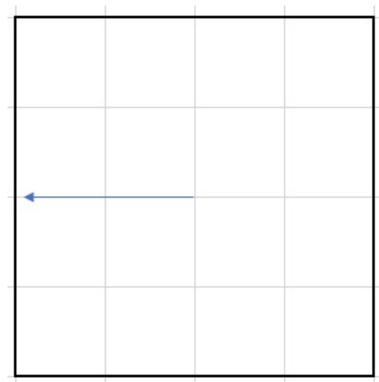
$$t_S = \frac{\sqrt{2}L}{v} = \sqrt{2}t_0.$$

How long does it take the teacher to get down there:

$$t_T = \frac{4L}{3v} = \frac{4}{3}t_0 < 1.4t_0 < \sqrt{2}t_0 = t_S.$$

So the teacher is there before the student arrives, and the student does not escape.

(b) *Straight Approach* In this approach the student heads to the nearest point to them (that is heading away from the teacher):

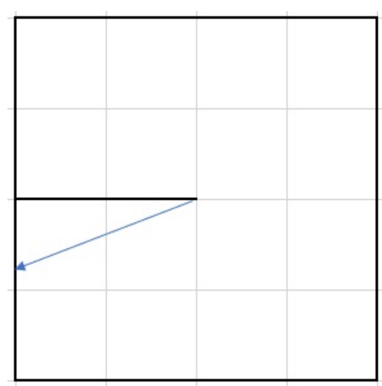


The student travels a distance L while the teacher travels a distance $3L$:

$$t_S = \frac{L}{v} = t_0$$
$$t_T = \frac{3L}{3v} = t_0 = t_S,$$

the teacher arrives at the side at exactly the same point as the student, and so the student cannot escape.

(c) *Angled Approach* In this approach the student heads off at an angle away from the teacher:



The distance s_S that the student travels satisfies:

$$\cos\theta = \frac{L}{s_S} \implies s_S = \frac{L}{\cos\theta} = \sec\theta L,$$

and therefore

$$t_S = \frac{L\sec\theta}{v} = \sec\theta t_0.$$

The teacher goes a distance $3L$ plus a little distance d that satisfies:

$$\tan\theta = \frac{d}{L} \implies d = L\tan\theta.$$

Therefore

$$t_T = \frac{3L + L\tan\theta}{3v} = \left(1 + \frac{\tan\theta}{3}\right) t_0.$$

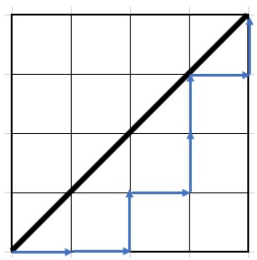
Recall for $\theta = 0$ the teacher catches the student. Similarly for $\theta = 45°$. What about at $\theta = 22.5°$? Then

$$t_S = \sec(22.5°)t_0 \approx 1.08t_0 < 1.14t_0 = \left(1 + \frac{\tan(22.5°)}{3}\right) = t_T.$$

In fact it is possible to show that for all $0 < \theta < \tan^{-1}(3/4)$, that is $0 < \theta \lesssim 36.87°$, this approach will work.

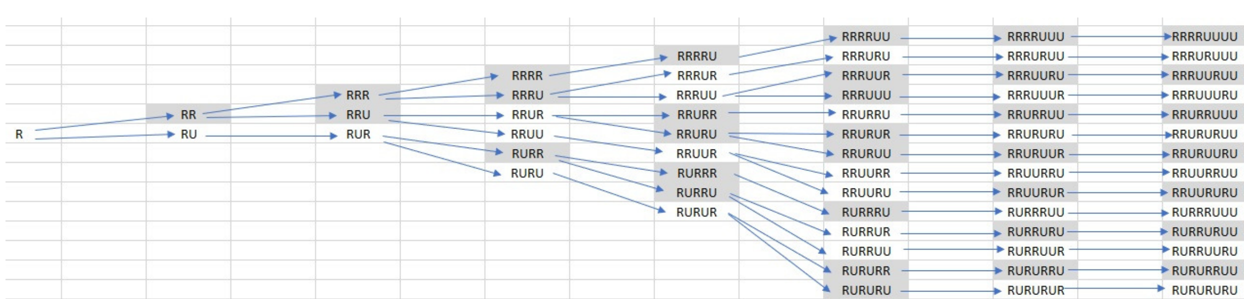
It is probably good news that the student can escape — it is a much harder proposition to show that out of the infinite number of strategies, that none work.

9. The n-th *Catalan number* is equal to the number of *monotonic lattice paths* along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Here is an example of a monotonic lattice path which does not pass above the diagonal in a 4×4 grid:



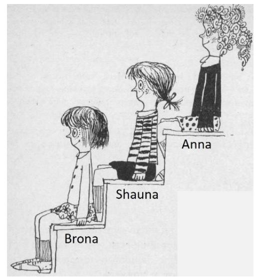
Find the fourth Catalan number by finding the number of such paths in a 4×4 grid.

Solution: There are various ways of answering this question but for $n = 4$ it is hard to beat brute force:



R is going right, and U is going up. There cannot be at any stage more U than R, and each sequence must have four R and four Us. There are 14 ways; the answer is 14.

10. Three friends, Anna, Brona and Shauna, are seated in a lecture hall. Shauna can only see Brona, Anna can see both girls seated in front of her.



In a box there are 2 white hats and 3 black hats. Three hats are taken out of the box and are put on the three girls. The girls did not see what hats they were given and do not know the color of the hats left in the box. When Anna was asked about the color of her hat she said she could not answer. When Shauna heard Anna's answer, she also said that she could not figure out the color of her hat. Can Brona, based on the other girls' answers, figure out the color of the hat she is wearing?

Solution: Some notation: (x, y, z) signifies the hats of Brona, Shauna, Annas respectively. Let us start with Anna. If the hat setup is (W, W, z), then Anna knows because there are two white hats only, that she must be wearing black. She does not answer, and so the state is (B, W, z), (W, B, z), or (B, B, z).

Shauna knows these must be the only possibilities. If she saw that Brona had a white hat, then she would not that she, Shauna herself, would be wearing a black hat. However she must see Brona in a black hat, and so she is stuck between (B, W, z) and (B, B, z).

Knowing that Brona is stuck between (B, W, z) and (B, B, z), Brona knows that she is wearing a black hat.